A Simplified Variation of Parameters Approach to Euler's Equations

D. L. Richardson[1] **and J. W. Mitchell**[1]

A normalized form of Euler's equations is rewritten in a variation of parameters approach using amplitudes and angular displacement as parameters. This new form is compact and yields a more accurate numerically integrated solution over longer simulation times than does a conventional integration of the Euler equations.

1 Introduction

Variational methods have played an important role in solving ordinary differential equations since their introduction by John Bernoulli in the late 17th century. Perhaps the most famous application of a variational method, the variation of orbital elements or the variation of constants, was performed by Leonhard Euler between 1748 and 1752 to describe the mutual perturbations of Jupiter and Saturn. However, it was not until 1782 that Joseph-Louis Lagrange fully and completely developed the method of the variation of parameters (VOP) in an application involving cometary motion. His approach is widely used today, particularly in astrodynamics applications where perturbed two-body motion is considered.

Because the moment-free rigid-body motion equations developed by Euler in 1754 admit an analytical solution, it seemed desirable to apply Lagrange's method of the variation of parameters to these equations. It was hoped that the resulting differential equations would enjoy the same numerical robustness that characterizes

[1] Department of Aerospace Engineering and Engineering Mechanics, University of Cincinnati, Cincinnati, OH 45221-0070.

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the VOP astrodynamics equations. We have indeed found this to be the case through a careful selection of VOP system parameters.

A VOP approach to Euler's equations and to the general orientational motion of rigid bodies has been investigated several times since the mid-1970s. Early work by Kraige and Junkins (1976) and Donaldson and Jezewski (1977) used the body's kinetic energy and angular momentum as the primary parameters in a general VOP scheme. The resulting equations were algebraically complex. The later work of Kraige and Skaar (1977) and Kraige (1978) did introduce less complexity to the form of the variational equations. More recently, Bond (1996) investigated a VOP scheme for rigid-body motion that utilized the case of symmetric, torque-free motion as the analytical basis for generating the parameters. Taken altogether, we felt that there was still room for improvement through a more judicious selection of parameters developed from the analytical solution to the general problem of torque-free rotation.

In the following, we present our development of Euler's equations in a VOP form using parameters that we feel are more natural and intrinsic to the problem. Our resulting VOP equations are not algebraically complex making them desirable for use in numerical simulations. We apply our approach to the constant-torque problem to demonstrate how our VOP equations provide greater accuracy integrations over longer simulation periods independent of the numerical propagation method chosen.

2 Formulation

2.1 Analytical Solution. The classical Euler's equations for rigid-body motion are given by the coupled first-order system

$$\left.\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + M_1, \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_1\omega_3 + M_2, \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 + M_3. \end{aligned}\right\} \tag{1}$$

The ω_i are the components of the angular velocity of the body expressed in a body-fixed principal-axis coordinate frame. The I_i and M_i are the principal-axis moments of inertia and external moment components, respectively.

Rewriting Eqs. (1), we have

$$\left.\begin{aligned} \dot{\omega}_1 &= -A_1\omega_2\omega_3 + M_1/I_1, \\ \dot{\omega}_2 &= A_2\omega_1\omega_3 + M_2/I_2, \\ \dot{\omega}_3 &= -A_3\omega_1\omega_2 + M_3/I_3, \end{aligned}\right\} \tag{2}$$

where

$$\left.\begin{aligned} A_1 &= (I_3 - I_2)/I_1, \\ A_2 &= (I_3 - I_1)/I_2, \\ A_3 &= (I_2 - I_1)/I_3, \end{aligned}\right\} \tag{3}$$

and $I_3 > I_2 > I_1$.

Upon performing a change of variables defined by the relations

$$\Omega_i = \omega_i/\sqrt{A_i} \quad \text{and} \quad \frac{d\tau}{dt} = \sqrt{A_1A_2A_3}\,, \tag{4}$$

the Euler equations become the normalized expressions

$$\left.\begin{aligned} \Omega_1' &= -\Omega_2\Omega_3 + G_1, \\ \Omega_2' &= \Omega_1\Omega_3 + G_2, \\ \Omega_3' &= -\Omega_1\Omega_2 + G_3, \end{aligned}\right\} \tag{5}$$

where the prime indicates $d/d\tau$ and G_i are the rescaled external moments.

Two independent integrals of motion, c_1 and c_2, of the unperturbed equations are obtained from the expressions

$$\left.\begin{aligned} \frac{d}{d\tau}\left(\frac{1}{2}\Omega_1^2\right) &= \Omega_1\Omega_1' = -\Omega_1\Omega_2\Omega_3, \\ \frac{d}{d\tau}\left(\frac{1}{2}\Omega_2^2\right) &= \Omega_2\Omega_2' = \Omega_1\Omega_2\Omega_3, \\ \frac{d}{d\tau}\left(\frac{1}{2}\Omega_3^2\right) &= \Omega_3\Omega_3' = -\Omega_1\Omega_2\Omega_3. \end{aligned}\right\} \tag{6}$$

and, by inspection, are seen to be

$$\left.\begin{aligned} \Omega_1^2 + \Omega_2^2 &= c_1^2, \\ \Omega_2^2 + \Omega_3^2 &= c_2^2. \end{aligned}\right\} \tag{7}$$

These integrals of motion are related to the classical integrals of energy T and angular momentum H by

$$c_1^2 = \frac{2TI_3 - H^2}{(I_3 - I_1)(I_3 - I_2)} \geq 0, \tag{8}$$

$$c_2^2 = \frac{H^2 - 2TI_1}{(I_3 - I_1)(I_2 - I_1)} \geq 0. \tag{9}$$

The solutions for the Ω_i of the unperturbed system are developed by first considering the second of Eqs. (5). Combining this with Eqs. (7) gives

$$\Omega_2' \triangleq c_1c_2\sqrt{1 - s^2}\,\sqrt{1 - k^2s^2}\,, \tag{10}$$

with $s = \Omega_2/c_1$ and $k = c_1/c_2 \geq 0$ where k is called the modulus. By restricting k such that $0 \leq k < 1$ and integrating Eq. (10), we obtain

$$\mathrm{sn}^{-1}(s) = c_2(\tau - \tau_0) \triangleq u. \tag{11}$$

Using this result along with Eqs. (7) produces the solutions for the Ω_i:

$$\left.\begin{aligned} \Omega_1 &= c_1\mathrm{cn}(u), \\ \Omega_2 &= c_1\mathrm{sn}(u), \\ \Omega_3 &= c_2\mathrm{dn}(u) \end{aligned}\right\} \tag{12}$$

It is seen that c_1, c_2, and τ_0 form a natural set of elements (parameters) for the characterization of the solution to the moment-free Euler equations. The variable u is viewed as the intrinsic angular variable for this problem.

2.2 Variation of Parameters. The variations of the two independent integrals of motion, c_1 and c_2, are obtained by direct differentiation of Eqs. (7) along with substitutions from Eqs. (5), yielding

$$\left.\begin{aligned} c_1' &= \frac{1}{c_1}(\Omega_1G_1 + \Omega_2G_2) = \mathrm{cn}(u)G_1 + \mathrm{sn}(u)G_2, \\ c_2' &= \frac{1}{c_2}(\Omega_2G_2 + \Omega_3G_3) = k\,\mathrm{sn}(u)G_2 + \mathrm{dn}(u)G_3. \end{aligned}\right\} \tag{13}$$

From a numerical perspective, integrating directly for the argument u is somewhat more algebraically desirable than integrating the variational equation for τ_0 and then forming u. Accordingly, we develop u' directly by manipulation of Ω_1 and Ω_2 from Eqs. (12) to produce

$$\mathrm{cn}(u)\Omega_2' - \mathrm{sn}(u)\Omega_1' = c_1u'\mathrm{dn}(u)$$
$$- \frac{c_1\mathrm{dn}(u)}{kk_c^2}[E(u) - k_c^2u - k^2\mathrm{sn}(u)\mathrm{cn}(u)/\mathrm{dn}(u)]k'. \tag{14}$$

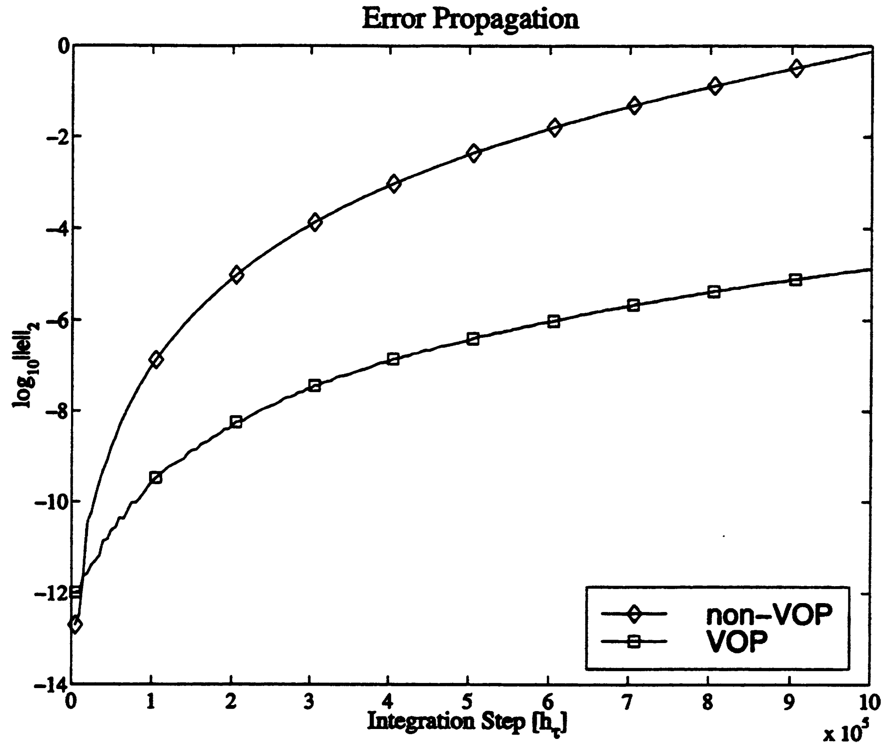

Fig. 1 Two-norm errors for numerical integrations of the modified Euler equations (non-VOP) and the variation of parameters equations (VOP) for approximately 3000 cycles of Ω_3

In the above expression, $k_c^2 = 1 - k^2$ is the complementary modulus, $E(u)$ is the incomplete elliptic integral of the second kind, and k' is given by

$$k' = \frac{1}{c_2}[\mathrm{cn}(u)G_1 + k_c^2\mathrm{sn}(u)G_2 - k\,\mathrm{dn}(u)G_3]. \quad (15)$$

After some algebraic manipulation, we arrive at the desired expression

$$u' = \frac{1}{c_1 k_c^2}[k(c_2^2 - c_1^2) - \mathrm{sn}(u)\mathrm{dn}(u)G_1 + k_c^2\mathrm{cn}(u)\mathrm{dn}(u)G_2 + k^3\mathrm{sn}(u)\mathrm{cn}(u)G_3 + [E(u) - k_c^2 u][\mathrm{cn}(u)G_1 + k_c^2\mathrm{sn}(u)G_2 - k\,\mathrm{dn}(u)G_3]]. \quad (16)$$

In summary, the variation of parameters solution to the perturbed Euler system of normalized Eqs. (5) is given by Eqs. (12) with the differential equations for the parameters given by Eqs. (13) and (16).

3 A Numerical Demonstration

We demonstrate the numerical behavior of our variation of parameters solution by comparisons to numerically generated solutions of the normalized Euler system of Eqs. (5). The solution to the problem of a constant torque applied to the body is of interest in satellite attitude determination applications (Williams and Tanygin, 1996). We chose this problem to illustrate the robustness of our VOP formulation. To this end, we selected the external moment components G_i to be constants of the same order of magnitude as the Ω_i components. We used

$$G_1 = G_2 = G_3 = 1.0. \quad (17)$$

We completed our system initialization by specifying values at $\tau = 0$ as

$$c_1(0) = 0.5, \quad c_2(0) = 1.0, \quad u(0) = 0.0, \quad (18)$$

which produced the initial conditions

$$\Omega_1(0) = c_1, \quad \Omega_2(0) = 0.0, \quad \Omega_3(0) = c_2. \quad (19)$$

Using these initial conditions, the variational equations (Eqs. (12) through (16)) and Euler's equations (Eqs. (5)) were integrated numerically using a Runge-Kutta 4/5 scheme. Both systems of equations were integrated with the same fixed step size. This step size h was taken to be approximately 1/400th of the initial $\mathrm{dn}(u)$ period as specified by the initial conditions, i.e., $h = K/200$, where $K \equiv K(k)$ is the complete elliptic integral of the first kind. In this case, we have $K(0.5) \approx 1.686$.

Both sets of double precision (IEEE Standard 754) integrations were compared to an extended precision (16-byte) integration of Eqs. (5). The extended precision step size was approximately 1/4000th of the initial $\mathrm{dn}(u)$ period. This integration was performed by using a 16th-degree first-order Chebyshev procedure (Richardson et al., 1998).

For each integration, we computed and graphed the two-norm error,

$$e = \sqrt{(\Delta\Omega_1)^2 + (\Delta\Omega_2)^2 + (\Delta\Omega_3)^2}, \quad (20)$$

where $\Delta\Omega_i = \Omega_i^* - \Omega_i$ is the difference of Ω_i when compared with the extended precision result, Ω_i^*. Figure 1 shows the error propagation results.

From this figure, it is seen that the variation of parameters solution maintains an accuracy of several orders of magnitude better than that produced by direct integration of the normalized Euler equations. This behavior was expected largely because of similar behaviors that have been observed in the numerical integration of various sets of VOP equations used in astrodynamics applications.

4 Concluding Remarks

In this paper, a variation of parameters solution is developed for a normalized form of the arbitrarily perturbed Euler's equations for attitude motion. The three parameters of the variation, two amplitudes, and an angular displacement lead to a very compact form that yields higher numerical accuracy over longer simulation intervals than did direct integration of Euler's equations for the same fixed step size. This improved accuracy is the result of the reformulation using the variation of parameters and thus independent of the numerical integration scheme.

While the accuracy of the integration did improve, one disadvantage of this method is additional computational overhead. Much of this overhead is due to the calculation of the necessary special functions. This additional computation is consistent with the variation of parameters integrations of the orbital elements in astrodynamics applications and is expected. For applications involving algebraically complex moment components, the variational equations will produce comparable execution timings.

With regard to step size, the reader should note that the normalized Euler equations and the variation of parameters equations both contain the same spectral content. Consequently, the variation of parameters equations cannot be integrated using step sizes that are substantially greater than that needed for the normalized equations.

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