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AAS/AIAA Space Flight Mechanics Meeting

San Antonio, Texas 27-30 January 2002

AAS Publications Office, P.O. Box 28130, San Diego, CA 92198

**FOR PUBLIC
RELEASE**

ASC-02-0215 31-Jan-2002

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Abstract

From a Lagrangian approach for the development of the relative motion equations for a nonlinear Hill's problem, it is shown that the influence of a spherical primary mass takes the form of a third-body-like disturbing function expressed relative to the origin of Hill's rotating frame. The resulting Lagrangian and corresponding equations of motion are compact and can provide an easily obtainable representation of the nonlinear contributions to the motion to an arbitrary order using recursion relations. The relative-motion equations are expanded through third-order in the local Hill's coordinates and a correspondingly accurate successive approximations solution is developed to describe nonlinear periodic motions in the Hill's frame.

INTRODUCTION

Considerable attention has been given to the dynamics and control of formation flying satellites [1, 10, 6]. In particular, much effort has been given to determining the relative effects of perturbations on satellite formations. Frequently, preliminary mission analyses begin with what are commonly known as Hill's [4] or Clohessy-Wiltshire [3] equations. These are the linearized equations describing the relative motion of two bodies under the gravitational influence of a point-mass central body. Traditionally, these equations have been used to describe rendezvous maneuvers and typically prove useful only for a few orbital revolutions. More in-depth analyses have included the effects of central body oblateness, solar and electromagnetic perturbations, drag [11]; and eccentricity effects [5]. Consequently, it seems of interest to develop an analytical solution describing periodic motions in Hill's frame containing the most significant nonlinear contributions, which influence the relative motion as much as the effects of more well-known perturbations.

In this paper, we describe a third-order analytical perturbation solution developed by successive approximations that provides a very accurate periodic solution to a set of nonlinear Hill's equations.

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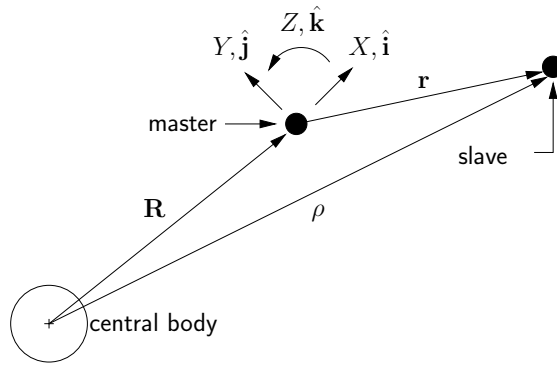


Figure 1: Master-slave geometry with $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$.

THIRD-ORDER NONLINEAR HILL'S EQUATIONS

The drag-free, nonlinear Hill's equations describing the motion of a *slave* relative to a *master* in a circular orbit of radius R and mean motion n are obtained from the relative-motion Lagrangian

$$L = \frac{1}{2} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \frac{GE}{\rho} + n^2 R X, \quad (1)$$

where

$$\mathbf{r} = X \hat{\mathbf{i}} + Y \hat{\mathbf{j}} + Z \hat{\mathbf{k}},$$

and

$$\rho^2 = R^2 + r^2 + 2 \mathbf{r} \cdot \mathbf{R}.$$

X, Y, Z form a local dextral system of coordinates with origin at the master as shown in Figure 1. The X axis ($\hat{\mathbf{i}}$) lies in the direction of the radial line outward from the central body (Earth), Y is oriented in the along-track (velocity) direction ($\hat{\mathbf{j}}$) and Z lies along the orbit normal ($\hat{\mathbf{k}}$) in the direction of the angular momentum.

It is algebraically convenient to introduce nondimensional coordinates and a nondimensional independent variable τ via the equations

$$X = R x, \quad Y = R y, \quad Z = R z, \quad \frac{d}{dt} = n \frac{d}{d\tau}. \quad (2)$$

The new Lagrangian for the motion is the scaled Lagrangian $L \rightarrow \mathcal{L} = L/(n^2 R^2)$ transformed from Eq. (1) along with the substitution, $GE = n^2 R^3$, obtained from Kepler's Third Law. The result is

$$\mathcal{L} = \frac{1}{2} \varrho' \cdot \varrho' + \frac{1}{\sqrt{1 + 2x + \varrho^2}} + x. \quad (3)$$

The square root contribution is recognized as the generating function for the Legendre polynomials $P_n(-x/\varrho) = (-1)^n P_n(x/\varrho)$ having argument $-x/\varrho$ where ϱ is the scaled distance from the master to the slave with

$$\varrho = x \hat{\mathbf{i}} + y \hat{\mathbf{j}} + z \hat{\mathbf{k}}.$$

This produces the infinite series expansion

$$\mathcal{L} = \frac{1}{2} \varrho' \cdot \varrho' + \sum_{n \geq 2} (-\varrho)^n P_n(x/\varrho). \quad (4)$$

Using the above Lagrangian, the equations of motion for the slave relative to the master are obtained through third-order in the nondimensional coordinates. We find

$$\left. \begin{aligned} x'' - 2y' - 3x &= -\frac{3}{2}(2x^2 - y^2 - z^2) + 2x(2x^2 - 3y^2 - 3z^2), \\ y'' + 2x' &= 3xy - \frac{3}{2}y(4x^2 - y^2 - z^2), \\ z'' + z &= 3xz - \frac{3}{2}z(4x^2 - y^2 - z^2). \end{aligned} \right\} \quad (5)$$

Linear Solution and Constraints

If the nonlinear contributions on the right-hand sides of Eqs. (4) are ignored, we have the well-known set of linear Hill's equations that are often used as the basis for rendezvous and formation flying analyses:

$$\left. \begin{aligned} x'' - 2y' - 3x &= 0, \\ y'' + 2x' &= 0, \\ z'' + z &= 0. \end{aligned} \right\} \quad (6)$$

The characteristic equation for the x, y system is

$$\lambda^2 (\lambda^2 + 1) = 0.$$

The repeated roots, $\lambda = \{0, 0\}$, give rise to a linear solution that contains a secular drift. We write the general solution as

$$\left. \begin{aligned} x(\tau) &= a_1 + a_3 \cos \tau + a_4 \sin \tau, \\ y(\tau) &= a_2 - \frac{3}{2}a_1\tau + 2a_4 \cos \tau - 2a_3 \sin \tau, \\ z(\tau) &= a_5 \cos \tau + a_6 \sin \tau. \end{aligned} \right\} \quad (7)$$

The integration constants a_i in the general solution are related to the components of the initial state vector $(x_0, y_0, z_0, x'_0, y'_0, z'_0)^T$ through the relations

$$\left. \begin{aligned} a_1 &= 4x_0 + 2y'_0, \\ a_2 &= y_0 - 2x'_0, \\ a_3 &= -3x_0 - 2y'_0, \\ a_4 &= x'_0, \\ a_5 &= z_0, \\ a_6 &= z'_0. \end{aligned} \right\} \quad (8)$$

We impose the following constraints on the initial conditions to insure that the linear solution is periodic about the origin of coordinates:

$$a_1 = 4x_0 + 2y'_0 = 0, \quad (9)$$

$$a_2 = y_0 - 2x'_0 = 0. \quad (10)$$

The algebra encountered in our nonlinear developments is simplified somewhat if we rewrite the periodic solution to Eqs. (6) in phase/amplitude form. To this end, we have

$$\left. \begin{aligned} x(\tau) &= -A \cos(\tau + \phi), \\ y(\tau) &= 2A \sin(\tau + \phi), \\ z(\tau) &= B \sin(\tau + \psi). \end{aligned} \right\} \quad (11)$$

NONLINEAR APPROXIMATION

Since the periodic Hill's problem is the relative difference in motion of two gravitationally non-interacting two-body motions with identical semimajor axes, we do not expect the frequencies of our nonlinear periodic solution to be a function of the amplitudes of the motion. Nor do we expect to have to impose any equality constraints on the amplitudes of motion as was required in a similar successive approximations derivation for periodic motions about the collinear points in the circular-restricted problem [8, 9]. Nevertheless, to provide additional algebraic flexibility, we introduce a frequency parameter ω and once again change the independent variable from τ to s via the transformation

$$\frac{d}{d\tau} = \omega \frac{d}{ds}. \quad (12)$$

The successive approximations solution to the third-order equations of motion given in Eqs. (5) are the asymptotic series

$$\left. \begin{aligned} x(s) &= x_1 + x_2 + x_3 + \mathcal{O}(4), \\ y(s) &= y_1 + y_2 + y_3 + \mathcal{O}(4), \\ z(s) &= z_1 + z_2 + z_3 + \mathcal{O}(4). \end{aligned} \right\} \quad (13)$$

The system frequency ω is assumed to have an asymptotic development of

$$\omega = 1 + \omega_1 + \omega_2 + \mathcal{O}(3). \quad (14)$$

The subscripted functions above are taken to obey the order of magnitude relations

$$x_i, y_i = \mathcal{O}(A^i), \quad z_i = \mathcal{O}(B^i), \quad \omega_i = \mathcal{O}(A^i). \quad (15)$$

To fourth-order, the equations of motion are rewritten and separated into the equations of the first-, second- and third-order. For the first-order, we have

$$\left. \begin{aligned} x_1'' - 2y_1' - 3x_1 &= 0, \\ y_1'' + 2x_1' &= 0, \\ z_1'' + z_1 &= 0. \end{aligned} \right\} \quad (16)$$

In these equations and those following, primes denote d/ds . From Eqs. (11), the solution is

$$\left. \begin{aligned} x_1 &= -A \cos(s + \phi), \\ y_1 &= 2A \sin(s + \phi), \\ z_1 &= B \sin(s + \psi). \end{aligned} \right\} \quad (17)$$

The equations for the second- and third-order are

$$\left. \begin{aligned} x_2'' - 2y_2' - 3x_2 &= -2\omega_1(x_1'' - y_1') - \frac{3}{2}(2x_1^2 - y_1^2 - z_1^2), \\ y_2'' + 2x_2' &= -2\omega_1(y_1'' + x_1') + 3x_1y_1, \\ z_2'' + z_2 &= -2\omega_1z_1'' + 3x_1z_1, \end{aligned} \right\} \quad (18)$$

and

$$\left. \begin{aligned} x_3'' - 2y_3' - 3x_3 &= -2\omega_1(x_2'' - y_2') - 2\omega_2(x_1'' - y_1') - \omega_1^2x_1'' \\ &\quad - 3(2x_1x_2 - y_1y_2 - z_1z_2) + 2x_1(2x_1^2 - 3y_1^2 - 3z_1^2), \\ y_3'' + 2x_3' &= -2\omega_1(y_2'' + x_2') - 2\omega_2(y_1'' + x_1') - \omega_1^2y_1'' \\ &\quad + 3(x_1y_2 + x_2y_1) - \frac{3}{2}y_1(4x_1^2 - y_1^2 - z_1^2), \\ z_3'' + z_3 &= -2\omega_1z_2'' - (2\omega_2 + \omega_1^2)z_1'' + 3(x_1z_2 + x_2z_1) \\ &\quad - \frac{3}{2}z_1(4x_1^2 - y_1^2 - z_1^2), \end{aligned} \right\} \quad (19)$$

respectively.

Second-Order Corrections

To eliminate secular terms in the particular solutions to Eqs. (18), the (expected) assignment

$$\omega_1 = 0 \quad (20)$$

is required. After substituting from Eqs. (17), the equations for the second-order corrections are

$$\left. \begin{aligned} x_2'' - 2y_2' - 3x_2 &= \frac{3}{4}(2A^2 + B^2) - \frac{9}{2}A^2 \cos 2u - \frac{3}{4}B^2 \cos 2v, \\ y_2'' + 2x_2' &= -3A^2 \sin 2u, \\ z_2'' + z_2 &= -\frac{3}{2}AB[\sin(u+v) + \sin(v-u)], \end{aligned} \right\} \quad (21)$$

where the abbreviations in the trigonometric arguments are

$$u = s + \phi, \quad v = s + \psi.$$

Constructing particular solutions is algebraically straight forward. The results are

$$\left. \begin{aligned} x_2 &= -\frac{1}{4}(2A^2 + B^2) + \frac{1}{2}A^2 \cos 2u + \frac{1}{4}B^2 \cos 2v, \\ y_2 &= \frac{1}{4}A^2 \sin 2u - \frac{1}{4}B^2 \sin 2v, \\ z_2 &= \frac{1}{2}AB[\sin(u+v) - 3\sin(v-u)]. \end{aligned} \right\} \quad (22)$$

Third-Order Corrections

Substituting the lower-order solutions into the right-hand sides of Eqs. (19) produces

$$\left. \begin{aligned}
 x_3'' - 2y_3' - 3x_3 &= \frac{3}{4}AB^2 \cos(u - 2v) + A \left(\frac{9}{4}A^2 + 2\omega_2 \right) \cos u \\
 &\quad - \frac{3}{4}AB^2 \cos(u + 2v) - \frac{25}{4}A^3 \cos 3u, \\
 y_3'' + 2x_3' &= -\frac{3}{8}AB^2 \sin(u - 2v) + A \left(\frac{9}{8}A^2 + 2\omega_2 \right) \sin u \\
 &\quad + \frac{3}{8}AB^2 \sin(u + 2v) - \frac{39}{8}A^3 \sin 3u, \\
 z_3'' + z_3 &= 2B\omega_2 \sin v - 3A^2B \sin(2u + v).
 \end{aligned} \right\} \quad (23)$$

It is seen that the only choice for removing the $\sin v$ term in the z_3'' equation is to impose the condition

$$\omega_2 = 0. \quad (24)$$

Accordingly, the frequency corrections are all zero to this order of analysis, as expected. Integrating to find a particular solution for z_3 produces

$$z_3 = \frac{3}{8}A^2B \sin(2u + v). \quad (25)$$

It would seem that there are various terms in the coupled x_3'' , y_3'' equations that could contribute secular results to the solution. However, this is not the case. If the y_3'' equation is integrated once and the result substituted into the x_3'' equation, we have

$$x_3'' + x_3 = -AB^2 \cos(u + 2v) - 3A^3 \cos 3u,$$

with the particular solution

$$x_3 = \frac{1}{8}AB^2 \cos(u + 2v) + \frac{3}{8}A^3 \cos 3u. \quad (26)$$

This result allows us to write an expression for y_3' which is integrated to give

$$y_3 = -\frac{1}{8}AB^2 \sin(u + 2v) + \frac{7}{24}A^3 \sin 3u + \frac{3}{8}AB^2 \sin(u - 2v) - \frac{9}{8}A^3 \sin u. \quad (27)$$

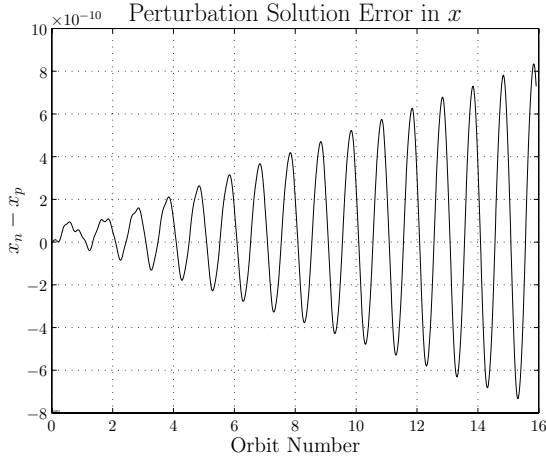


Figure 2: x -direction comparison of perturbation solution to integration of third-order differential equations.

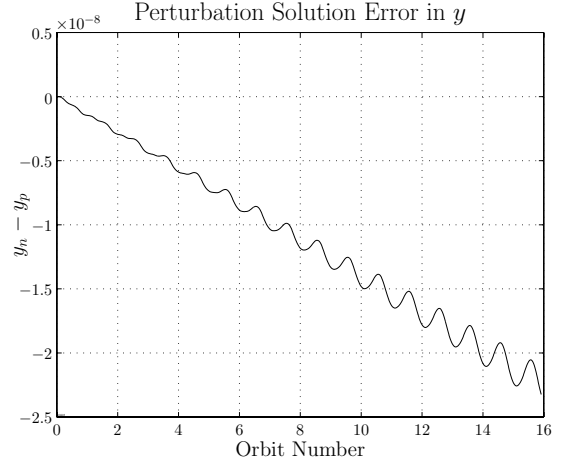


Figure 3: y -direction comparison of perturbation solution to integration of third-order differential equations.

COMPLETE SOLUTION AND NUMERICAL TESTING

We collate the above results and write our third-order successive approximations solution to Eqs. (5) as follows:

$$\left. \begin{aligned}
 x(\tau) &= -A \cos u - \frac{1}{4} (2A^2 + B^2) + \frac{1}{2} A^2 \cos 2u + \frac{1}{4} B^2 \cos 2v \\
 &\quad + \frac{1}{8} AB^2 \cos(u + 2v) + \frac{3}{8} A^3 \cos 3u, \\
 y(\tau) &= 2A \sin u + \frac{1}{4} A^2 \sin 2u - \frac{1}{4} B^2 \sin 2v - \frac{1}{8} AB^2 \sin(u + 2v) + \frac{7}{24} A^3 \sin 3u \\
 &\quad + \frac{3}{8} AB^2 \sin(u - 2v) - \frac{9}{8} A^3 \sin u, \\
 z(\tau) &= B \sin v + \frac{1}{2} AB [\sin(u + v) - 3 \sin(v - u)] + \frac{3}{8} A^2 B \sin(2u + v),
 \end{aligned} \right\} (28)$$

where

$$u = \tau + \phi, \quad v = \tau + \psi$$

because $\omega = 1$, thus making $s = \tau$.

Having developed a perturbation solution, we compared this to a numerical integration of the third-order nonlinear differential equations given by Eqs. (5). The fixed stepsize numerical integration was performed using a Runge-Kutta 4/5 method [2] spanning approximately one day at an orbital altitude of 500 km.

The initial conditions were selected by specifying the unscaled amplitudes and phases used in the perturbation solution. We chose

$$A^* = 20 \text{ km}, \quad B^* = 4 \text{ km}, \quad \phi = 0, \quad \psi = \pi/2. \quad (29)$$

The appropriately scaled amplitudes A, B and ϕ, ψ were then used with Eqs. (28) along with their derivatives to compute the test set of initial conditions.

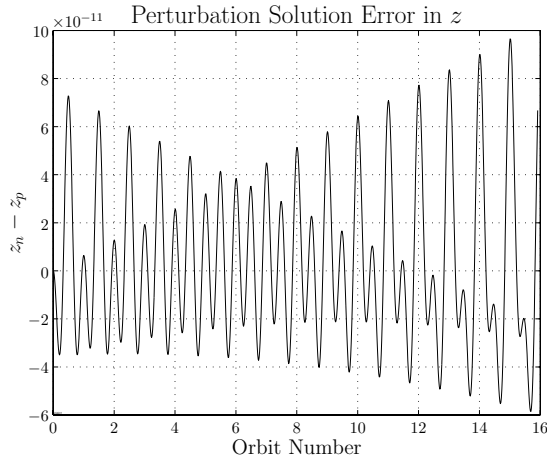


Figure 4: z -direction comparison of perturbation solution to integration of third-order differential equations.

diff	amplitude	avg
x	4.1052×10^{-06}	$+4.3961 \times 10^{-06}$
y	2.2746×10^{-06}	-4.7845×10^{-09}
z	1.0596×10^{-06}	-3.1647×10^{-06}

Table 1: Perturbation to linear solution comparison.

The position difference between the third-order perturbation solution and the numerical integration can be seen in Figures 2–4. Overall, the perturbation solution appears to represent a periodic solution to the actual third-order equations. The accuracy remained consistent with the truncation errors in the successive approximations solution. There is a small along-track secular runoff seen in Figure 3. The likely source for this behavior is attributed to a small difference between the semimajor axes of the slave and the master. This discrepancy creates a small difference in orbit periods which over a one day simulation interval is seen as a small along-track separation. For orbital configurations of similar energies, this drift will remain small on the order of the truncation errors in the development of the equations of motion. All this is reinforced in Figure 3.

For comparison, Table 1 shows the difference between the linear solution and the perturbation solution. Since the differences are harmonic, the amplitude and average of the differences between the linear and perturbation solution are provided. The amplitude of the oscillation indicates that the differences are small, but not insignificant as a perturbing effect.

CONCLUSIONS

In this paper, a third-order successive approximations solution describing relative motion in the neighborhood of a circular reference orbit was presented. The equations of mo-

tion were developed from a compact relative-motion Lagrangian for two gravitationally non-interacting satellites. The resulting analytical solution compared well to a numerical integration of the third-order nonlinear differential equations over a simulation period of one day. The perturbation solution errors in the radial, along-track, and orbit normal directions remained on the order of millimeters, centimeters, and tenths of millimeters, respectively, throughout. These differences remained small even in the presence of slight differences in orbital energies. Discrepancies were consistent with local truncation errors.

ACKNOWLEDGEMENTS

This worked was performed while the second named author held a National Research Council Research Associateship Award at the Air Force Research Laboratory (AFRL) in the Air Vehicles Directorate Control Theory Optimization Branch (VACA) located at the Wright-Patterson Air Force Base.

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