

ORBIT DETERMINATION ACCURACY REQUIREMENTS FOR COLLISION AVOIDANCE

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A simulation has been developed to examine how orbit determination accuracy, size of avoidance maneuvers, encounter geometry and warning time affect the probability of collision. The simulation shows that reasonably sized collision avoidance maneuvers are effective in reducing the probability of collision only if accurate orbit determination data is available for both the active satellite and the threat. Also, a method to determine the maneuvers required to reduce the probability of collision to a desired value is presented.

INTRODUCTION

A collision between satellites in a constellation and other objects in space could potentially destroy one, many or possibly even all satellites in the constellation. In order to build an effective collision avoidance system for such a constellation, trade studies must be performed to understand the sensitivity of the probability of collision (P_c) to various characteristics of the system. The purpose of this paper is to present the orbit determination (OD) accuracy required for an effective collision avoidance system for a commercial satellite constellation. In addition, this paper will present methods for predicting potential collisions between satellites, computing the probability of collision and determining orbital maneuvers to avoid those collisions

GENERIC COLLISION AVOIDANCE SIMULATOR

The Generic Collision Avoidance Simulator (GCAS) was developed to support analysis of collision avoidance requirements. Some mathematical simplifications were made in the formulation of GCAS to allow very fast run times to enable exploration of the entire design trade space. The following assumptions are made in GCAS:

1. GCAS simulates two satellites in near-circular orbits at the same inclination and semi-major axis, but with different values of RAAN ($1 \text{ deg} \leq \Delta\Omega \leq 359 \text{ deg}$). All motion is described by the restricted two body equations.
2. Maneuvers performed by active satellites produce a change in the close approach distance in only the in-track direction. No maneuvers to change inclination or RAAN are considered and only small changes in mean semi-major axis ($< 20 \text{ m}$) are considered. The change in radial and cross-track distance due to any collision avoidance maneuver is small compared to the in-track change.

GCAS simulates close approaches by computing the orbit elements necessary for a given miss distance to occur at the point of closest approach. These orbit elements are converted to J2000 Earth Centered Inertial (ECI) state vectors and fed into the probability of collision calculation, which is described in the Appendix. The user may vary the initial covariance matrices, the difference in RAAN between the two objects, the miss distance, and the prediction time span to determine the effects on the probability of collision.

Though orbit determination performance is usually quoted in terms of position error and velocity error, a more meaningful error criteria is the semi-major axis error. Semi-major axis error propagated in time drives the in-track position uncertainty. This can be readily seen by analyzing the two body orbit equation for mean anomaly. In the two body problem, all orbit elements are constant except for mean anomaly, which changes as:

$$M = M_0 + \sqrt{\frac{\mu}{a^3}}(t - t_0) \quad (1)$$

Since all of the orbit elements are constant except for M , the state transition matrix (for the orbit elements) is the identity matrix for all times, except for one element:

$$\Phi_{6,1} = \frac{\partial M}{\partial a_0} = -\frac{3}{2} \sqrt{\frac{\mu}{a_0^5}} (t - t_0) \quad (2)$$

Now suppose we can estimate the semi-major axis such that its error variance is σ_a^2 . To determine how the error in our estimate of mean anomaly grows in time, we form the product of $\Phi P \Phi^T$, to find that:

$$\sigma_M^2(t) = \sigma_M^2(t_0) + \frac{9\mu}{4a_0^5} (t - t_0)^2 \sigma_a^2 \quad (3)$$

Suppose that mean anomaly is initially known perfectly ($\sigma_M^2(t_0) = 0$), then

$$\sigma_M(t) = \frac{3}{2} \sqrt{\frac{\mu}{a_0^5}} (t - t_0) \sigma_a \quad (4)$$

In order to determine how intrack position uncertainty grows in terms of meters or kilometers for a near circular orbit, we multiply through by the semi-major axis to find:

$$\sigma_{intrack}(t) = \frac{3}{2} \sqrt{\frac{\mu}{a_0^3}} (t - t_0) \sigma_a \quad (5)$$

For a circular orbit the semi-major axis sigma can be found from the UVW covariance matrix using²:

$$\sigma_a^2 = 4\sigma_u^2 + 8a^2 \frac{v}{\mu} \sigma_{uv} + 4a^4 \frac{v^2}{\mu^2} \sigma_v^2 \quad (6)$$

where,

$$v = \sqrt{\mu/r}$$

The orbital period is $T = 2\pi a^{1.5} / \sqrt{\mu}$. Plugging these terms into our semi-major axis sigma equation results in the following equation:

$$\sigma_a = 2\sqrt{\sigma_u^2 + \frac{T}{\pi} \rho_{uv} \sigma_u \sigma_v + \frac{T^2}{4\pi^2} \sigma_v^2} \quad (7)$$

where ρ_{uv} is the correlation coefficient between radial position error and intrack velocity error.

For this study, we have defined four levels of orbit determination accuracy, corresponding to the accuracy of the data we expect from them. The performance characteristics of the four categories of orbit determination are shown below in Table 1.

Table 1 - PERFORMANCE CHARACTERISTICS OF VARIOUS ORBIT DETERMINATION SYSTEMS

OD SOURCE	3-SIGMA POSITION ERROR (M) AT EPOCH	ERROR GROWTH RATE (M/DAY)	SEMI-MAJOR AXIS SIGMA (M)	COMMENTS
Internal Ephemeris	30	200	1.5	
Improved Internal Ephemeris	20	20	0.15	
Laser Tracking	10	700	5	
Radar Tracking	500 – 1000	5,000 – 15,000	37	Uses General Perturbations Mean Elements

SIMULATION RESULTS

GCAS was used to simulate two satellites in near collision orbits being tracked by different orbit determination systems as shown below in Table 2.

Table 2 – TEST CASES

TEST CASE	PRIMARY OBJECT OD QUALITY	SECONDARY OBJECT OD QUALITY	COLLISION PREDICTION TIME (DAYS)
A	Internal Ephemeris	Radar Tracking	1
B	Internal Ephemeris	Laser Tracking	1
C	Improved Internal Ephemeris	Laser Tracking	1
D	Internal Ephemeris	Radar Tracking	2
E	Internal Ephemeris	Laser Tracking	2
F	Internal Ephemeris	Laser Tracking	2

The intrack miss distance was varied from 0 to 12 km to examine the effect of intrack maneuvers within the entire extent of a +/-6.0-km control box. The collision geometry was also varied as described below.

Effects of Geometry

The geometry of the close approach has a large effect on the probability of collision. In order to examine how geometry affects collision probability, GCAS was run to simulate two satellites in near-circular orbits at the same inclination and semi-major axis, but with different values of RAAN ($1 \text{ deg} \leq \Delta\Omega \leq 359 \text{ deg}$).

The probability of collision was reported at 1-degree intervals of $\Delta\Omega$. Then the satellites were moved by various amounts in the intrack direction so that they ended up with intrack miss distances between 0 and 1 km.

Figure 1 shows how the probability of collision changes when the geometry of the close approach ($\Delta\Omega$) is changed between the two orbits. The blue curve (0 km miss distance) simulates two satellites estimated to be on a collision course. Note that the collision probability reaches a sharp peak when the two satellites collide head-on. The other curves are generated by moving the satellites by various amounts in the intrack direction from a collision (0.1, 0.3 and 0.5 km respectively). It is interesting to note that the largest decrease in collision probability is achieved near $\Delta\Omega = 180$ so that only a small maneuver is required to achieve a given collision probability, when compared with other angles. This plot shows the probability of collision that would be predicted two days in advance of a collision, using Internal Ephemeris as the source of data for one satellite and Laser Tracking as the source for the other.

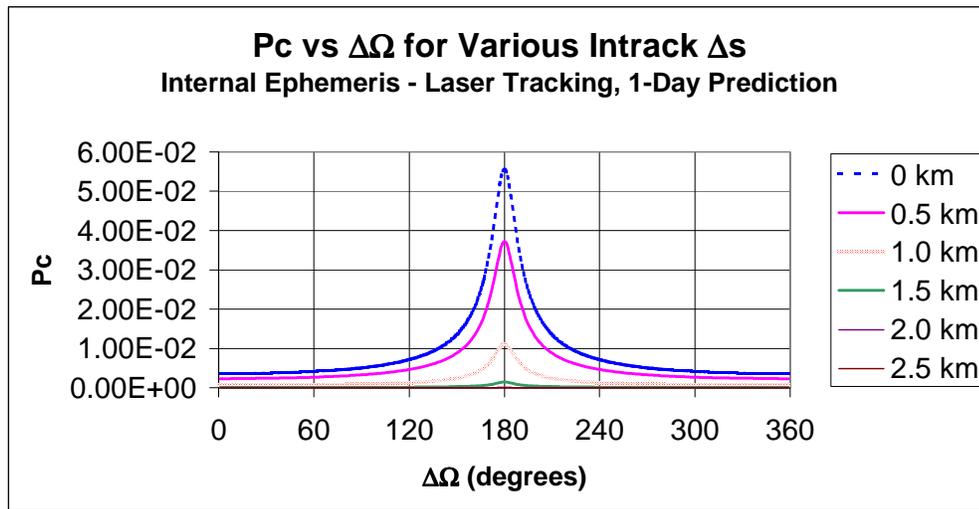


Figure 1 - Variations in Probability of Collision Due to Collision Geometry

Effects of Orbit Determination Quality

In order to examine how the estimated probability of collision is affected by orbit determination quality, GCAS was used to simulate satellites that were tracked by the orbit determination systems listed in Table 1.

Figure 2 shows how collision probability varies with intrack miss distance for various OD qualities, using a one-day prediction time. Note that in this case, there is very little reduction in collision probability estimated using Radar Tracking data even after a 6-km intrack maneuver. So there is little to be gained from a maneuver if you are using Radar Tracking data for collision prediction because the uncertainty is too large. On the other hand, a 2-km intrack maneuver is required when using Laser Tracking data to reduce the probability below that predicted when using Radar Tracking data.

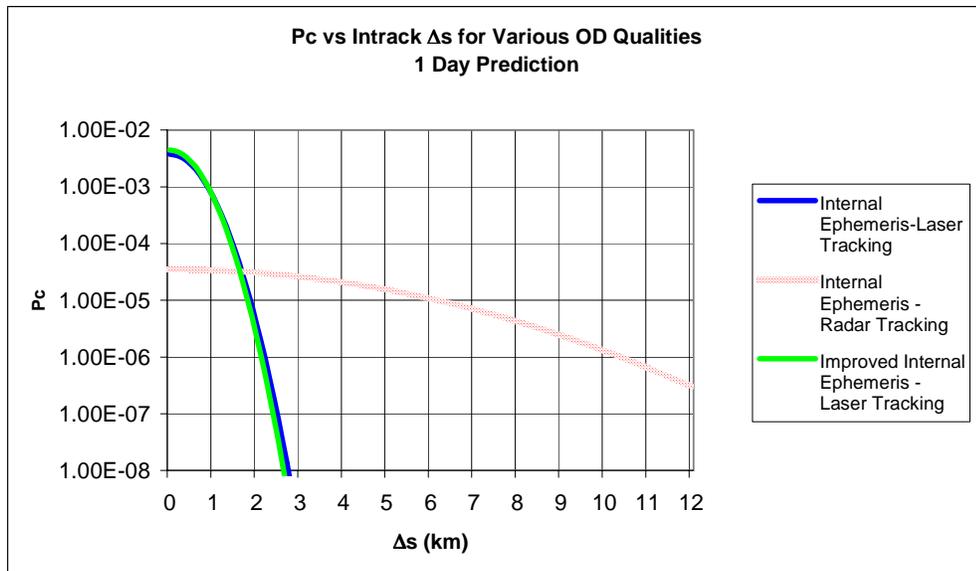


Figure 2 - Collision Probability vs Miss Distance For Various OD Qualities

Effects of Prediction Time

In Figure 3, the effects of 1- and 2- day prediction times on collision probability is examined. In the case of using Radar Tracking data, using a 1-day prediction span does not improve collision avoidance performance. However, when using Laser Tracking data, for a 1-day prediction span, the intrack change needed to get to a given P_c is about half of that for a 2-day span. However, the required intrack change for either a 1-day or 2-day prediction time can be achieved with the same change in semi-major axis.

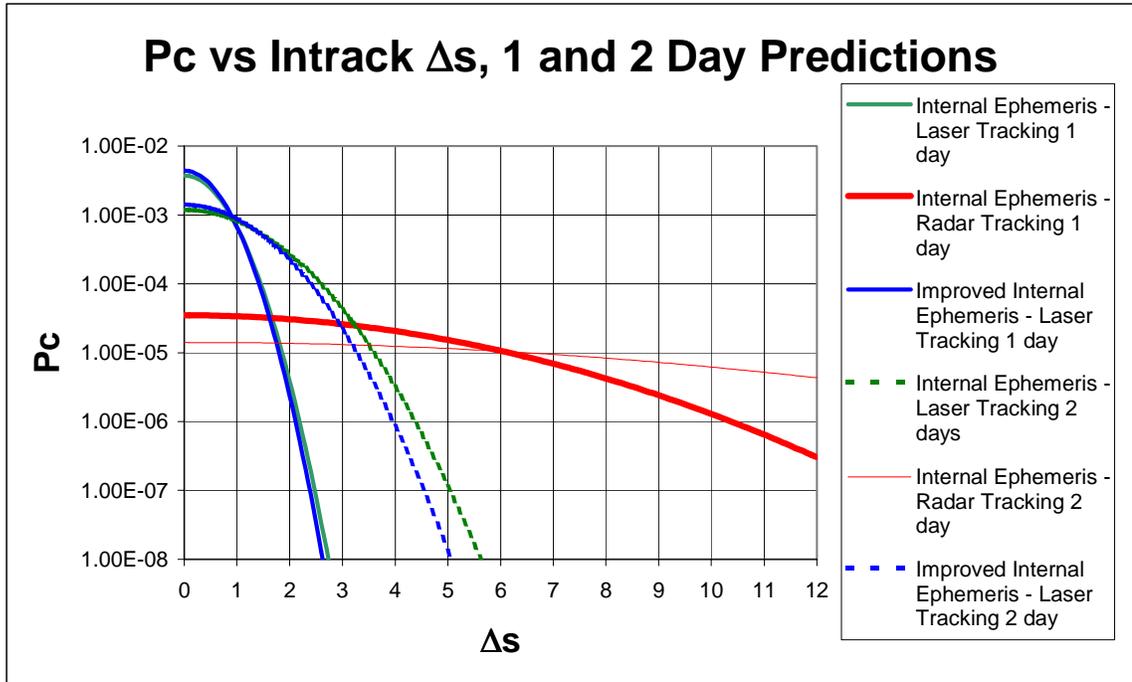


Figure 3 - Effects of Prediction Time and OD Quality on Collision Probability

Table 3 summarizes the results by presenting the intrack miss distance required to achieve a desired collision probability of $1.0E-06$ for each encounter using various combinations of orbit determination data and a 1-day prediction time. The smaller the intrack miss distance, the better. When considering that the maximum achievable intrack change per day (while staying within a +/- 6 km stationkeeping box) is 2.7 km, a design goal of requiring less than 1.0 km intrack change in a day to achieve a collision probability of less than $1.0E-06$ seems reasonable. However, the data in Table 3 shows that this is a very challenging design goal.

Table 3 – INTRACK MANEUVERS REQUIRED TO ACHIEVE $PC < 1.0E-06$

OD SOURCE	$\Delta\Omega = 31.587 \text{ DEG}$	$\Delta\Omega = 94.761 \text{ DEG}$	$\Delta\Omega = 157.935 \text{ DEG}$
Internal Ephemeris – Radar Tracking	10.4 km	10.9 km	12 km
Internal Ephemeris – Laser Tracking	2.2 km	1.3 km	2.4 km
Improved Internal Ephemeris – Laser Tracking	2.2 km	2.2 km	2.4 km
Design Goal	< 1.0 km	< 1.0 km	< 1.0 km

COLLISION AVOIDANCE MANEUVERS

Analytic Collision Prediction

The first step in the collision avoidance process is predicting potential satellite collisions. Orbit determination is performed for all of the subject satellites using any of several techniques, such as: ground based laser tracking, ground based RF tracking, onboard GPS receivers, etc. Once the orbits for all satellites of interest have been determined, the Analytic Collision Prediction Algorithm is used to determine which pairs of satellites may collide. Traditionally, this has been done by numerically integrating the equations of motion for each object. Then the position of each object would be compared to every other object at each time step over the time span of interest. This can be a very large job when a number of satellites are considered. The Analytic Collision Prediction Algorithm improves the speed and efficiency of collision prediction by employing simple geometric calculations instead of numerical integration of differential equations.

The Analytic Collision Prediction Algorithm can be used to:

1. Evaluate proposed orbital maneuvers for collision risk.
2. Evaluate the distance of closest approach for a given constellation design.
3. Determine locations in a given constellation where a satellite should not be placed.

The Analytic Collision Prediction Algorithm reduces the computational burden by analytically ruling out pairs of objects that cannot collide without integrating equations of motion. The Analytic Collision Prediction Algorithm is composed of two filters; the first filter is the Apogee-Perigee filter and the second is the Argument of Latitude filter. The Apogee-Perigee filter checks to see if the apogee of one object is below perigee of the other (with some tolerance to account for uncertainty) or vice versa. If so, this pair of objects will not collide.

The Argument of Latitude filter rapidly determines if collisions are possible between any two satellites of a group of satellites that have nearly the same semi-major axis but are in different planes. If a pair of satellites is predicted to potentially collide, their orbits are precisely computed using numerical integration to determine the actual distance of closest approach. The probability of collision between the two satellites is computed based on the precise orbits and the uncertainties in the orbital data. If the probability of collision is deemed to be too high, one of the two satellites is commanded to perform an orbital maneuver to reduce the probability of collision.

Once the Apogee-Perigee filter is passed, collisions can only occur near the line of intersections of the orbit planes. Any two non-coplanar orbit planes cross each other at two locations, which are located 180 degrees away from each other. Spherical trigonometry or vector analysis can be used to determine the locations of these crossings. Figure 4 below illustrates the geometry of two orbital planes crossing.

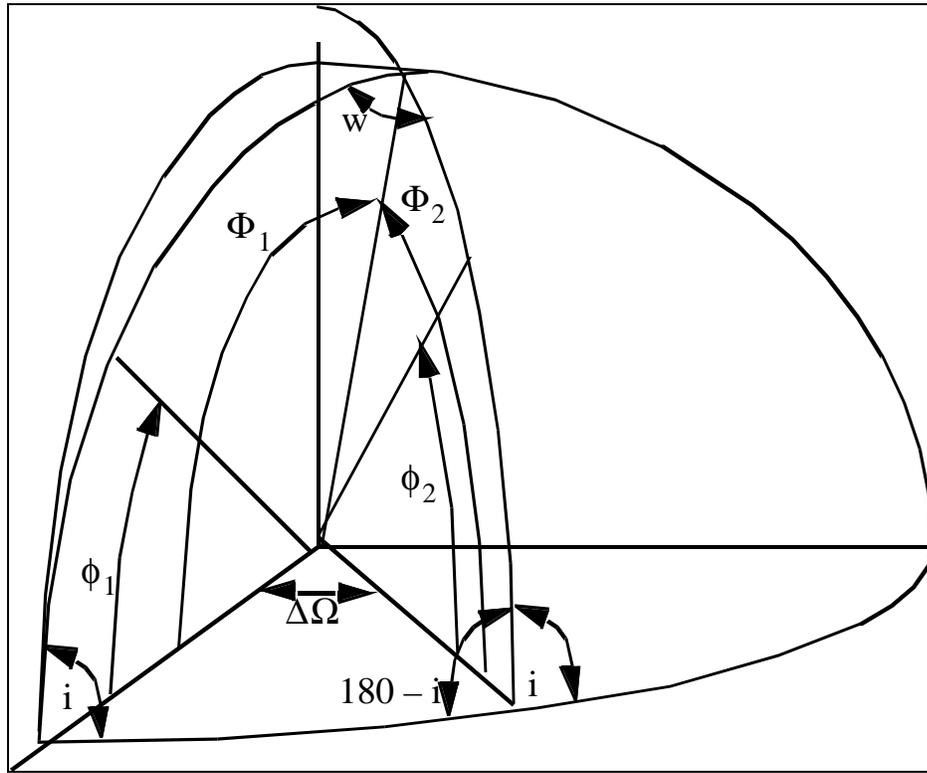


Figure 4 - Collision Geometry

Figure 4 illustrates the geometry of two orbital planes crossing and the important angular relationships. The inclination angle is represented by i . $\Delta\Omega$ represents the angle between the ascending nodes of the two planes. The incident angle or wedge angle between the planes is w . The angles Φ_1 and Φ_2 are defined as the angle between the equatorial plane and the point of the plane crossing measured in each of the orbital planes, or in other words, the argument-of-latitude of the plane crossing. The angles ϕ_1 and ϕ_2 represent the arguments-of-latitude of a satellite in each plane at a given time. The relative argument of latitude is defined to be $\delta\phi = \phi_1 - \phi_2$.

We can use the law of cosines to solve for the wedge angle, w by,

$$\cos w = -\cos i_1 \cos(180 - i_2) + \sin i_1 \sin(180 - i_2) \cos \Delta\Omega \quad (8)$$

Now, given w , we can use the law of sines to find the arguments-of-latitude of the plane crossing,

$$\frac{\sin w}{\sin \Delta\Omega} = \frac{\sin i_1}{\sin \Phi_2} = \frac{\sin(180 - i_2)}{\sin \Phi_1} \quad (9)$$

If $i_1 = i_2 = i$, this set of equations simplifies to,

$$\sin \Phi_2 = \sin \Phi_1 = \frac{\sin i \sin \Delta\Omega}{\sin w} \quad (10)$$

Upon first inspection it would seem that $\Phi_1 = \Phi_2$, however, $\Phi_1 = 180 - \Phi_2$ is also a valid solution and in fact, it can be shown that this solution is the correct one.

From Figure 4 above, it can be seen that a collision between satellites in two planes is possible when the following condition is true:

$$\delta\phi = \phi_1 - \phi_2 = \Phi_1 - \Phi_2 \quad (11)$$

This is defined to be the collision criterion. Also, since the values of Φ_1 and Φ_2 are constant, (assuming that nodal regression for both satellites is the same or at least slowly varying), it is convenient to define a new variable for the difference as,

$$\Delta\Phi_c = \Phi_1 - \Phi_2 \quad (12)$$

This means that if the difference between the arguments-of-latitude of any two satellites in different planes is near $\Delta\Phi_c$, a collision is possible. The collision criterion is the basis for our new close approach prediction method. Since the equation is so simple, it provides a very fast and efficient means for predicting close approaches.

Application 1: Maneuver Evaluation. A simple check can be devised to determine if a set of satellites performing intrack maneuvers is going to have a close approach. In order for a close approach to occur:

$$\delta\phi(t) = \phi_1(t) - \phi_2(t) = \Delta\Phi_c(i_1, i_2, \Delta\Omega) \quad (13)$$

Thus, a pair of satellites will have a close approach during a maneuver if one of the following conditions is true:

$$\phi_1(\text{initial}) - \phi_2(\text{initial}) = \Delta\Phi_c \quad (14)$$

or

$$\phi_1(\text{final}) - \phi_2(\text{final}) = \Delta\Phi_c \quad (15)$$

or

$$\phi_1(\text{initial}) - \phi_2(\text{initial}) < \Delta\Phi_c \quad \text{AND} \quad \phi_1(\text{final}) - \phi_2(\text{final}) > \Delta\Phi_c \quad (16)$$

or

$$\phi_1(\text{initial}) - \phi_2(\text{initial}) > \Delta\Phi_c \quad \text{AND} \quad \phi_1(\text{final}) - \phi_2(\text{final}) < \Delta\Phi_c \quad (17)$$

This says that for a given set of initial and final target arguments of latitude, if the difference between a pair of satellites in different planes starts on one side of $\Delta\Phi_c$, the difference must remain on the same side of $\Delta\Phi_c$ or else there will be a close approach. Once the initial and target arguments of latitude are determined, if one of the above conditions is met, the only way to avoid a close approach is to ensure that the satellites are at different altitudes at the plane crossing. Therefore, we can determine the number of close approach pairs for a given satellite maneuver scenario without computing the actual trajectories of the satellites.

Application 2: Constellation Design Evaluation. Given a set of initial angles, ϕ , for a particular constellation design, the pairs of satellites with the closest approaches can be determined using:

$$\phi_{ij}(\text{initial}) - \phi_{kl}(\text{initial}) - \Delta\Phi_c(\Delta p) = \Delta\lambda \quad (18)$$

$$s = r\Delta\lambda \quad (19)$$

where s is the close approach distance and r is the radial distance from the satellite to the center of the earth.

Application 3: Where Not to Place Satellites. In designing and operating a constellation, it is often useful to know the arguments of latitude in each plane could potentially be dangerous so operators can avoid placing spare satellites in those locations. By applying the collision criteria, we can determine these locations in each. In fact, for a constellation with all of the satellites at one particular altitude, each satellite creates a point in each of the other planes where an object would collide with that satellite. Therefore, for a constellation consisting of 126 operational satellites plus 14 spares, arranged in 14 planes, each plane would have 130 such points or danger zones. These danger zones can be mapped out for any given constellation and presented graphically as shown in Figure 5 below. Each point on the graph represents a dangerous location where a satellite would collide with a satellite in another plane in the constellation.

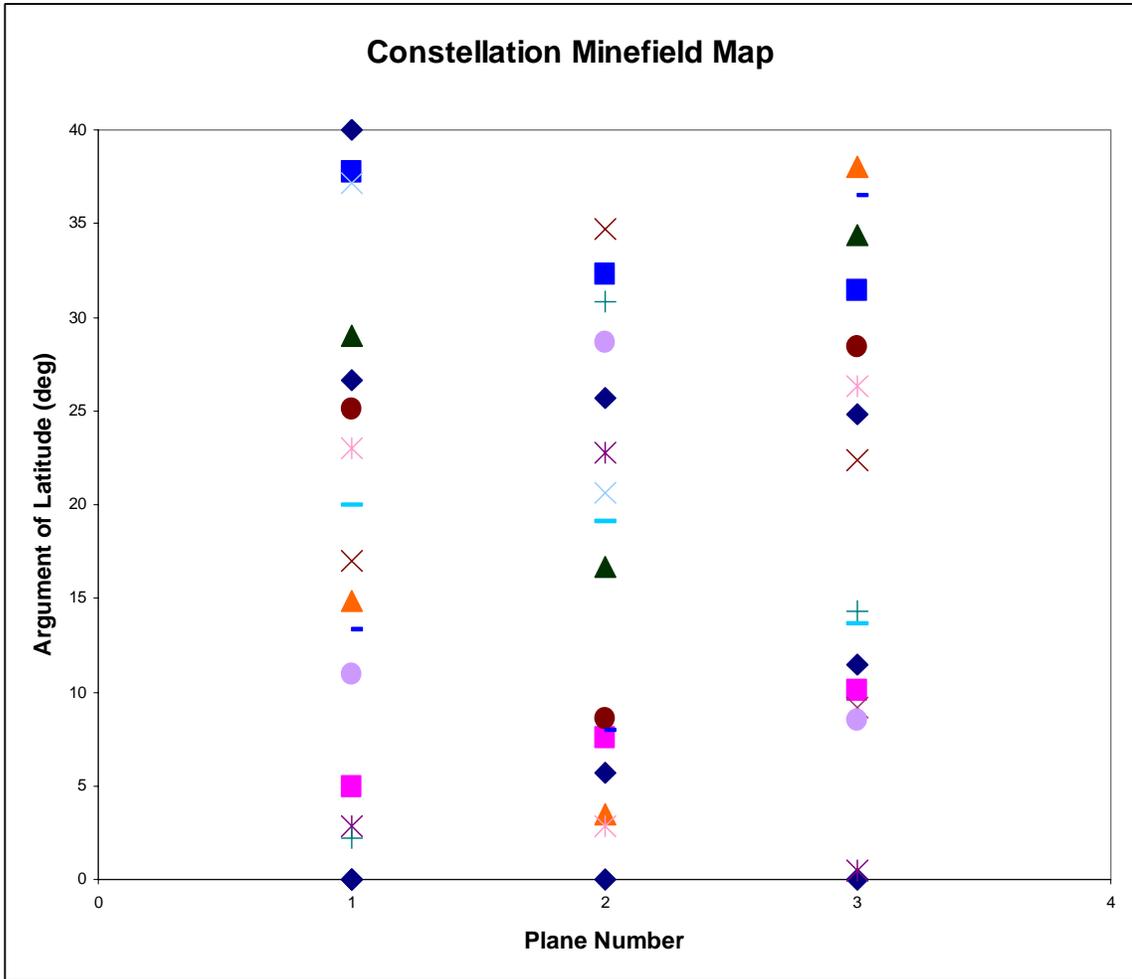


Figure 5 - Portion of a Constellation Minefield Map

Collision Avoidance Guidance

Once a close approach or collision has been predicted and a decision has been made to perform a maneuver to avoid the collision, Collision Avoidance Guidance is used to determine the appropriate orbital maneuver. Collision Avoidance Guidance is based on the collision criterion described above. Since a collision will occur if:

$$\Delta\phi(t) = \phi_1(t) - \phi_2(t) = \Delta\Phi_c(i_1, i_2, \Delta\Omega) \quad (20)$$

the objective is to not allow this condition to become true or if it does become true, the radial separation between the satellites at the orbital crossing points must be made large enough to achieve a low probability of collision. This radial separation can be achieved by an increase or decrease in semi-major axis or by changing the eccentricity vector or a combination of the two.

In order to understand how $\Delta\phi(t)$ behaves, a simple orbit model can be developed for each satellite:

$$\phi_i(t) = \phi_i(t_0) + \left(\sqrt{\frac{\mu}{(a_i(t_0) + \dot{a}_i t)^3}} \right) \cdot t, \text{ for } i = 1, 2 \quad (21)$$

where

μ = Earth gravity parameter

a = semi-major axis

\dot{a} = time rate of change of semi-major axis

t_0 = initial time

then

$$\Delta\phi(t) = \phi_1(t) - \phi_2(t) \quad (22)$$

Once $\Delta\phi(t)$ is computed for the time span of interest, it can be compared to $\Delta\Phi_c$ to determine the best maneuver.

For example, suppose two satellites are predicted to have a close approach. One of the satellites is a piece of space debris and the other is actively stationkeeping to make up for drag. Both satellites are in a circular orbit, with the space debris in a 10 meter higher orbit. Drag acts upon the space debris, bringing it closer to the live satellite. Then $\Delta\phi(t)$ would behave according to the blue line in Figure 6 below. $\Delta\phi(t)$ begins at some particular initial value and then since the semi-major axes of the two satellites are different, $\Delta\phi(t)$ increases for a time. Then as drag acts upon the debris object, its semi-major axis is brought down until they are equal. At this point, the direction of $\Delta\phi(t)$ turns around since the debris object is now below the active satellite.

Close approaches happen when the blue line touches or comes close to the red line, which represents $\Delta\Phi_c$. In this particular case, the intersection of the blue and red lines occurs at the maximum of the blue line which happens when the two semi-major axis values are equal. This is the worst case because the satellites are both at the same altitude at the time when $\Delta\phi(t)$ is at $\Delta\Phi_c$.

In order to avoid these predicted close approaches, the active satellite could perform a maneuver at t_0 to raise its orbit by 10 meters and avoid them. The green line in Figure 6 represents the post-maneuver trajectory. In this case, the satellite never has a close approach with the space debris because $\Delta\phi(t)$ is heading away from $\Delta\Phi_c$.

The size of the required maneuver is established by determining the miss distance required to achieve a desired probability of collision. The Collision Avoidance Guidance would find the minimum change in semi-major axis required to achieve the required miss distance.

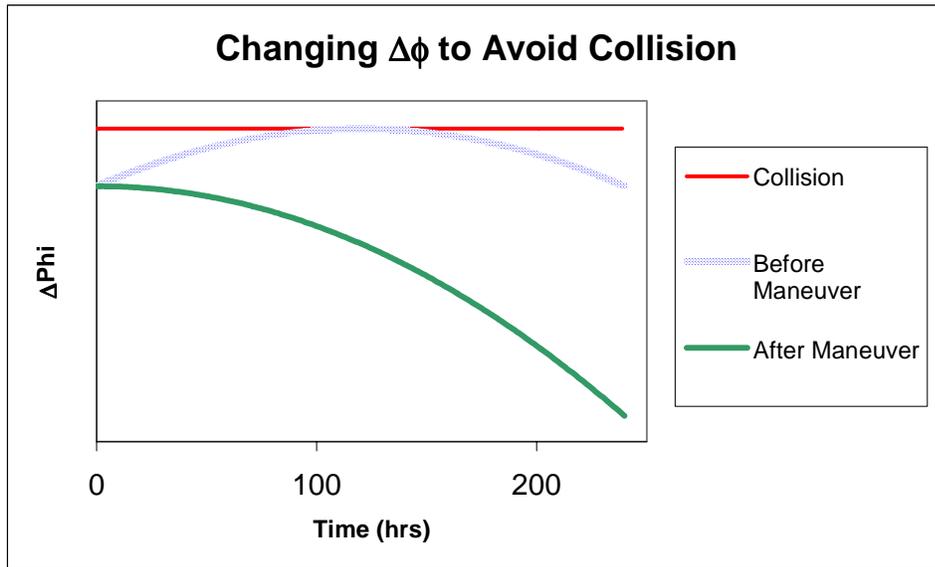


Figure 6 - Avoiding a Collision by Changing $\Delta\phi$

In some cases it may not be possible or desirable to maneuver the satellite in the direction needed to move $\Delta\phi(t)$ away from $\Delta\Phi_c$ for all time. The other option is the maneuver in the opposite direction. This will cause the close approach to happen sooner and cause a second close approach in the future as indicated by the purple line in Figure 7 below. However, these close approaches will happen when the semi-major axis values are not equal, causing them to miss by the amount of the semi-major axis difference. In this case, the Collision Avoidance Guidance would need to find the minimum maneuver to ensure that the miss distance in the radial direction is large enough to ensure that the desired probability of collision is achieved.

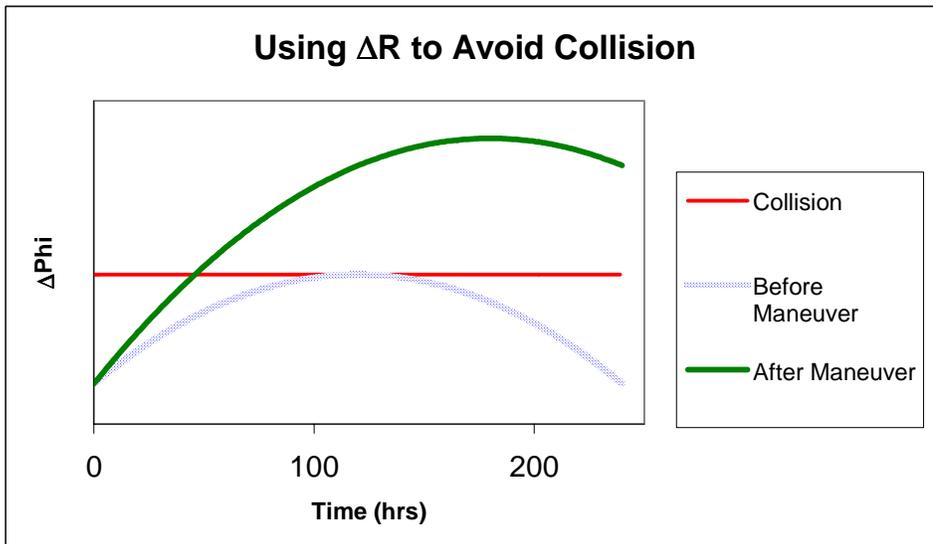


Figure 7 - Using Radial Position Difference to Avoid a Collision

So far only a weak tie has been established between Collision Avoidance Guidance and the probability of collision. However, this can be remedied by drawing contours of constant probability of collision in the collision phase plane. The collision phase plane is a two dimensional space consisting of the difference between the two satellites semi-major axes (or radius at the plane crossing point for non-circular orbits) along the y-axis and the difference between the relative argument of latitude, $\delta\phi$ and $\Delta\Phi_c$ along the x-axis. Ellipses of constant probability of collision can be determined by using the Probability of Collision Algorithm described in the Appendix. The ellipse corresponding to a chosen probability of collision threshold represents a keep out zone. The trajectory through this space can be modified by performing maneuvers to keep the trajectory outside of the keep out ellipse as shown in Figure 8 below. The dotted red line represents the keep out ellipse defined by a probability of collision of 0.00001. If the trajectory passes within this ellipse, the probability of collision will be at least 0.00001. However, by applying a 20-meter change in semi-major axis to one of the satellites, this region can be avoided (as shown by the green line), thus ensuring that the probability of collision will remain below 0.00001.

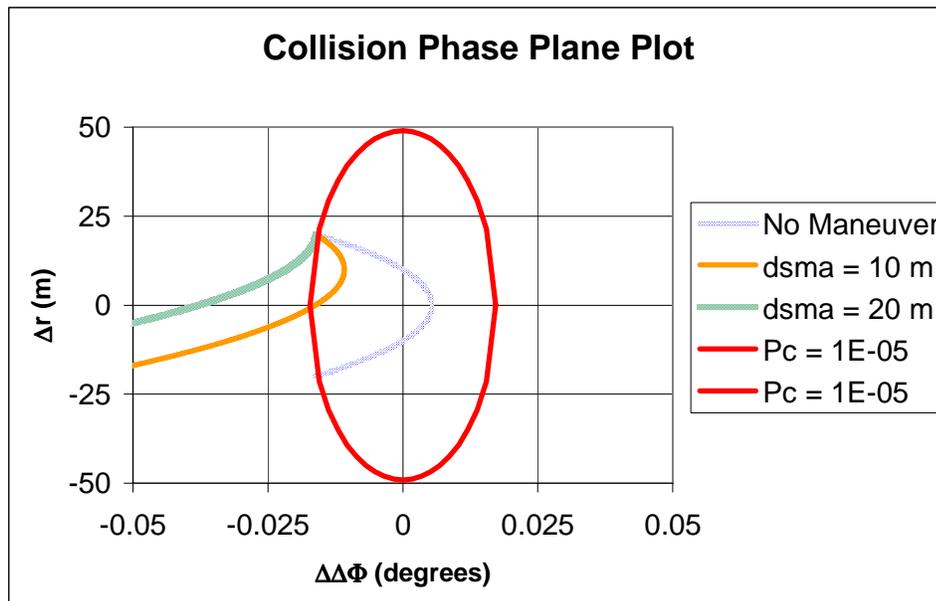


Figure 8 - Collision Phase Plane Plot

If it is possible for one of the satellites in question to perform a large out of plane maneuver, it may be desirable to consider an out of plane maneuver to avoid a collision. This is readily analyzed by extending the collision phase plane concept into the third dimension.

CONCLUSIONS

Effective collision avoidance is possible using only in-track maneuvers, but only if excellent orbit determination data for both the active SVs and threat objects is available. In order to achieve a significant reduction in the probability of collision, while remaining within a +/- 6 km stationkeeping box, at least Laser Tracking-quality data is required. It is clear that Radar Tracking data is not sufficiently accurate.

If excellent OD data is not available: DON'T MANEUVER. There is no appreciable reduction in the probability of collision from collision avoidance maneuvers unless OD is very good.

Geometry of the close approach is very important so the collision avoidance system needs to provide information that gives the user knowledge of the expected close approach geometry.

Collision prediction time spans of 1 or 2-days yield equivalent results in terms of achievable collision probability for a given change in semi-major axis (propellant). However, a 1-day prediction span is better

because the total change in position required to achieve a given collision probability is half of that needed when using a 2-day prediction time. This is especially important when the satellite is near the back of its stationkeeping box. Also, a 1-day prediction time allows less time for unmodeled effects or estimation errors to grow.

In addition, this paper has presented methods for predicting potential collisions between satellites, computing the probability of collision and determining orbital maneuvers to avoid those collisions, which are particularly useful for designing and operating large satellite constellations.

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APPENDIX – PROBABILITY OF COLLISION COMPUTATION

The method used to compute the probability of collision in GCAS is presented in this Appendix.

Suppose two satellites are going to have a close approach and we wish to calculate their probability of collision. Given a position and velocity vector and covariance matrix for each satellite expressed in Cartesian Earth Centered Inertial (ECI) coordinates $(\bar{r}_1, \bar{v}_1, C_1, \bar{r}_2, \bar{v}_2, C_2)$, an encounter coordinate system is defined with the Y axis pointed in the direction of relative velocity, so that a unit vector \hat{j} is defined as

$$\bar{v}_R = \bar{v}_1 - \bar{v}_2 \quad (23)$$

$$\hat{j} = \bar{v}_R / v_R \quad (24)$$

The direction of the other two unit vectors is arbitrary in the plane normal to \bar{v}_R . We may then compute the matrix that relates a vector in the encounter system to the ECI system by:

$$\bar{V}_{ECI} = M_R^{ECI} \bar{V}_R \quad (25)$$

The position of the first satellite in the encounter system is $(0, 0, 0)$ by definition and the position of the second satellite relative to the first in the ECI system is $\bar{r}_2 - \bar{r}_1$. Therefore, the position of the second satellite in the encounter system is:

$$\bar{r}_{R0} = (M_R^{ECI})^T (\bar{r}_2 - \bar{r}_1) \equiv \begin{bmatrix} x_{R0} \\ y_{R0} \\ z_{R0} \end{bmatrix} \quad (26)$$

However, since the relative motion is entirely in the Y direction, the only distance of importance is the distance in the XZ plane so a removal matrix, N , is defined so that:

$$\bar{\mu} = N \bar{r}_{R0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{R0} \\ y_{R0} \\ z_{R0} \end{bmatrix} \quad (27)$$

Since the orbits of the two satellites were estimated independently, the combined covariance matrix, Γ , is computed by:

$$\Gamma = N \left(M_R^{ECI} \right)^T [C_1 + C_2] \left(M_R^{ECI} \right) N^T \quad (28)$$

We may now compute the probability that the distance between the two satellites will be less than some collision radius, r , as the integral of the joint normal distribution function of the area, which is given by,

$$P_C = \frac{1}{2\pi\sqrt{|\Gamma|}} \int_{-r/2}^{r/2} \int_{-\sqrt{(r/2)^2 - z^2}}^{\sqrt{(r/2)^2 - z^2}} e^{-\frac{1}{2} \begin{bmatrix} x-\mu_x & z-\mu_z \end{bmatrix} \Gamma^{-1} \begin{bmatrix} x-\mu_x \\ z-\mu_z \end{bmatrix}} dx dz \quad (29)$$

If r is much less than the square root of the diagonal elements of Γ , this integral can be approximated by:

$$P_C \approx \frac{dx dz}{2\pi\sqrt{|\Gamma|}} e^{-\frac{1}{2} \begin{bmatrix} \mu_x & \mu_z \end{bmatrix} \Gamma^{-1} \begin{bmatrix} \mu_x \\ \mu_z \end{bmatrix}} = \frac{d^2}{2\pi\sqrt{|\Gamma|}} e^{-\frac{1}{2} \begin{bmatrix} \mu_x & \mu_z \end{bmatrix} \Gamma^{-1} \begin{bmatrix} \mu_x \\ \mu_z \end{bmatrix}} \quad (30)$$

However, if r is not much less than the square root of the diagonal elements of Γ , a more accurate but still simple approximation can be obtained as follows. Define a quantity Q as:

$$Q \equiv \begin{bmatrix} x-\mu_x & z-\mu_z \end{bmatrix} \Gamma^{-1} \begin{bmatrix} x-\mu_x \\ z-\mu_z \end{bmatrix} \equiv \begin{bmatrix} x-\mu_x & z-\mu_z \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x-\mu_x \\ z-\mu_z \end{bmatrix} \quad (31)$$

$$Q = a(x^2 - 2x\mu_x + \mu_x^2) + (b+c)(xz - x\mu_x - z\mu_x + \mu_x\mu_z) + d(z^2 - 2z\mu_z + \mu_z^2) \quad (32)$$

If we now perform a change of variables, such that $x = r \cos \theta$, $z = r \sin \theta$ and $\psi = b + c$, then we have

$$Q = r^2 (a \cos^2 \theta + \psi \sin \theta \cos \theta + d \sin^2 \theta) - r [(2a\mu_x + \psi\mu_z) \cos \theta + (\psi\mu_x + 2d\mu_z) \sin \theta] + a\mu_x^2 + \psi\mu_x\mu_z + d\mu_z^2 \quad (33)$$

Now if we define the following terms:

$$\Theta_1 = (2a\mu_x + \psi\mu_z) \cos \theta + (\psi\mu_x + 2d\mu_z) \sin \theta \quad (34)$$

$$\Theta_2 = (a \cos^2 \theta + \psi \sin \theta \cos \theta + d \sin^2 \theta) \quad (35)$$

$$q_0 = a\mu_x^2 + \psi\mu_x\mu_z + d\mu_z^2 \quad (36)$$

then

$$Q = r^2 \Theta_2 - r \Theta_1 + q_0 \quad (37)$$

Noting that Θ_1 and Θ_2 are purely functions of θ for a given covariance, the probability of collision can be written as,

$$P_C = \frac{1}{2\pi\sqrt{|\Gamma|}} \int_{Area} e^{-\frac{1}{2}Q} dA = \frac{e^{-\frac{1}{2}q_0}}{2\pi\sqrt{|\Gamma|}} \int_{Area} e^{-\frac{1}{2}(r^2\Theta_2 - r\Theta_1)} dA \quad (38)$$

We can approximate the evaluation of this integral by dividing the area of integration as shown below in Figure 9.

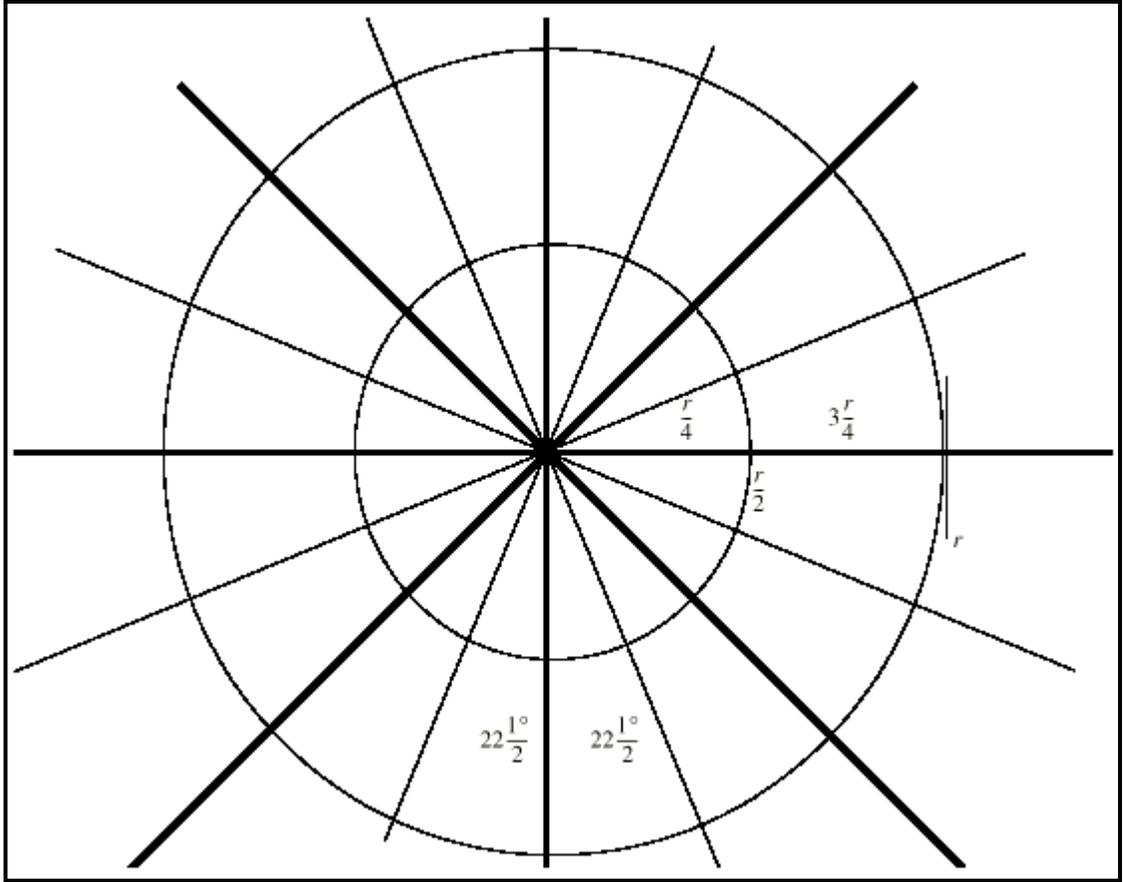


Figure 9 - Collision Probability Integration

Using Figure 9, we can approximate the integral by evaluating Θ_1 and Θ_2 at 0, 45, 90, 135, 180, 225, 270 and 315 degrees respectively. These need to be calculated only once for each covariance. Also, since the sines and cosines are known at these points, their values can just be placed in an array so that no time is wasted in their computation. Given these values, we can integrate over each of the eight pie-shaped pieces of area and then sum the results to arrive at the final probability of collision. Each area can be computed by,

$$\begin{aligned}
 P_C &\approx \frac{e^{-\frac{1}{2}q_0}}{2\pi\sqrt{|\Gamma|}} \sum_i \left(\frac{\pi}{4} \int_0^r \rho e^{-\frac{1}{2}(\rho^2\Theta_{2i} - \rho\Theta_{1i})} d\rho \right) = \frac{e^{-\frac{1}{2}q_0}}{2\pi\sqrt{|\Gamma|}} \sum_i \left(\frac{\pi}{4} \int_0^r \rho e^{-\frac{\rho^2\Theta_{2i}}{2}} e^{\frac{\rho\Theta_{1i}}{2}} d\rho \right) \\
 &= \frac{e^{-\frac{1}{2}q_0}}{8\sqrt{|\Gamma|}} \sum_i \left(\int_0^r \rho e^{-\frac{\rho^2\Theta_{2i}}{2}} e^{\frac{\rho\Theta_{1i}}{2}} d\rho \right)
 \end{aligned} \tag{39}$$

This integral is not in standard tables, however, it can be approximated by using some r in each part to evaluate the exponential and then multiplying by the area of that part and sum. The areas of the inner circle are all the same and are equal to $\pi r^2 / 32$. The areas of the outer annulus are all the same and are equal to

$\pi r^2/8 - \pi r^2/32 = 3\pi r^2/32$. The ρ to be used is $r/4$ in the inner wedges and $3r/4$ in the outer annuli. Therefore, the probability of collision can be written

$$P_C \approx \frac{e^{-\frac{1}{2}q_0}}{2\pi\sqrt{|\Gamma|}} \frac{\pi r^2}{32} \left(\sum_i e^{-\frac{1}{2}\left(\frac{r^2}{16}\Theta_{2i} - \frac{r}{4}\Theta_{1i}\right)} + 3 \sum_i e^{-\frac{1}{2}\left(\frac{9r^2}{16}\Theta_{2i} - \frac{3r}{4}\Theta_{1i}\right)} \right) \quad (40)$$

which reduces to

$$P_C \approx \frac{e^{-\frac{1}{2}q_0}}{2\pi\sqrt{|\Gamma|}} \frac{r^2}{32} \left(\sum_i e^{\frac{r}{8}\Theta_{1i} - \frac{r^2}{32}\Theta_{2i}} + 3 \sum_i e^{\frac{3r}{8}\Theta_{1i} - \frac{9r^2}{32}\Theta_{2i}} \right) \quad (41)$$

This results in a total of 17 exponential evaluations instead of one and should be more than adequate for most realistic covariance matrices and satellites.